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Abstract: In this paper, a universal formula is proposed for event-based stabilization of nonlinear time-delay systems affine in the control. The feedback is derived from the original one proposed by E. Sontag (1989) and previously extended to event-based control of nonlinear undelayed systems. Under the assumption of the existence of a control Lyapunov-Krasovsky functional, it enables smooth (except at the origin) asymptotic stabilization while ensuring that the sampling intervals do not contract to zero. Global asymptotic stability is obtain under the small control property assumption. Moreover, the control can be proved to be smooth anywhere under certain conditions. Some simulation results highlight the ability of the proposals.

Keywords: Event-based control, Nonlinear systems, Time delay, Stabilization.

INTRODUCTION

The classical (time-triggered) discrete time framework of controlled systems consists in sampling the system uniformly in time with a constant sampling period. This field has been widely investigated for linear systems, see Åström and Wittenmark (1997) and the references therein. In the case of nonlinear systems, one way to address a discrete-time feedback is to implement a continuous time control algorithm with a sufficiently small sampling period (this procedure is denoted as *emulation*). However, the hardware used to sample and hold the plant measurements or compute the feedback control action may make impossible the reduce of the sampling period to a level that guarantees acceptable closed-loop performance, as demonstrated in Hsu and Sastry (1987). Furthermore, although periodicity simplifies the design and analysis, it results in a conservative usage of resources since the control law is computed and updated at the same rate regardless it is really required or not. Some works hence recently addressed resource-aware implementations of the control law, where the control law is event-driven.

Other way to tackle the problem of discrete-time control for nonlinear systems is the application of sampled-data control algorithms based on an approximated discrete-time model of the process, like in Nešić and Teel (2004), which is not a trivial task. Another proposed approach consists in modifying a continuous time stabilizing control using a general formula to obtain a redesigned control suitable for sampled-data implementation, as done in Nešić and Grüne (2005). Some event-triggered control approaches have also been suggested as a solution to overcome such drawbacks of emulation, redesigned control and complexity of the underlying nonlinear sampled-data models.

The advantages of an event-based controller over a time-based one are mainly influenced by the way in which the event are generated. Typical event-based detection mechanisms are functions of the state variation (or the output) of the system, like in Årzén (1999); Sandee et al. (2005); Durand and Marchand (2009); Sánchez et al. (2009b,a). Although the event-triggered control is well-motivated and allows to relax the periodicity of computations, only few works report theoretical results about the stability, convergence and performance. In Åström and Bernhardsson (2002) in particular, it is proved that such an approach reduces the number of sampling instants for the same final performance. Some stability and robustness proprieties are exploited in Åström and Bernhardsson (2002); Heemels et al. (2009); Lunze and Lehmann (2010); Donkers and Heemels (2010); Eqtami et al. (2010). An alternative approach consists in taking events related to the variation of a Lyapunov function – and consequently to the state too – between the current state and its value at the last sampling, like in Velasco et al. (2009), or in taking events related to the time derivative of the Lyapunov function, like in Tabuada (2007); Anta and Tabuada (2008); Marchand et al. (2011, 2013). More particularly, the work in Marchand et al. (2013) is based on the universal formula of Sontag (1989). An event-based stabilization of general nonlinear systems affine in the input is proposed. The control updates ensure the strict decrease of a *control Lyapunov function* (CLF), and so is asymptotically stable the closed-loop system.

Eventually, the event-based control scheme is of growing interest in embedded and networked control systems (where the control loop is closed over a communication link) because it allows to reduce the sending/receiving of information needed for control and, consequently, a strong energy saving. A network has several advantages, like flexibility in the configuration of the communication structure and the number of systems. However, it also has a considerable impact on the performance, notably because of communication delays and packet losses which avoid real-time control constraints to be meet and can even cause the instability of the control loop. This is why it is also important to consider time-delays in the event-based approaches (note that the packet loss issue is not considered here). Only few works deal with this topic for linear systems, like in Lehmann and Lunze (2011, 2012); Guinaldo et al. (2012); Durand (2013). Nonetheless, in the best knowledge of the authors, this is the first time an event-based control strategy is proposed for general nonlinear time-delay systems.

The concept of CLF, which is a useful tool for designing robust control laws for nonlinear (undelayed) systems, has been extended to time-delay systems in the form of *control Lyapunov-Razumikhin functions* (CLRF) and *control Lyapunov-Krasovsky functionals* (CLKF), see Jankovic (1999, 2000, 2003). The latter form is more flexible and easier to construct than CLRFs. Moreover, if a CLKF is known for a nonlinear time-delay system, several stabilizing control laws can be constructed using one of the universal formulas derived for CLFs (such as the Sontag's formula for instance) to achieve global asymptotic stability of the closed loop system. In the present paper, we hence propose to extend the universal event-based formula of Marchand et al. (2013) for the stabilization of affine in the control nonlinear *time-delay* systems. The class of time-delay systems under consideration is restricted here to depend on some discrete delays and a distributed delay. Note also that we only consider state delays and do not consider delays in the control signal (input delays).

The rest of the document is organized as follows. In section 1, some definitions are introduced and the problem is stated. The main contribution is then presented in section 2. The smooth control particular case is also concerned and an example is depicted. An analysis finally concludes the paper.

1. PRELIMINARIES

1.1 Event-triggered stabilization of nonlinear systems

Let consider the general nonlinear dynamical system

$$\begin{aligned} &\dot{x}(t) = f\big(x(t), u(t)\big) \qquad (1)\\ &\text{with} \quad x(0) := x_0 \end{aligned}$$

with $x(t) \in \mathcal{X} \subset \mathbb{R}^p$, $u(t) \in \mathcal{U} \subset \mathbb{R}^q$ and f is a Lipschitz function vanishing at the origin. Note that only null stabilization is considered in this paper for the sake of simplicity, and the dependence on t can be omitted in the sequel. Also, let define $\mathcal{X}^* := \mathcal{X}\backslash\{0\}$ hereafter.

Definition 1.1. (Event-based feedback).
By *event-based feedback* we mean a set of two functions, that are ***i)*** *an event function* $\epsilon : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ that indicates if one needs (when $\epsilon \le 0$) or not (when $\epsilon > 0$) to recompute the control law and ***ii)*** *a feedback function* $\upsilon : \mathcal{X} \to \mathcal{U}$.

The solution of (1) with event-based feedback (ϵ, υ) starting in x_0 at $t = 0$ is then defined as the solution of the differential system

$$\dot{x}(t) = f\big(x(t), \upsilon(x_i)\big) \quad \forall t \in [t_i, t_{i+1}[\qquad (2)$$

$$\text{with} \quad x_i := x(t_i) \qquad (3)$$

where the time instants t_i, with $i \in \mathbb{N}$, are considered as *events* (they are determined when the event function ϵ vanishes and denote the sampling time instants) and x_i is the memory of the state value at the last event. With this formalization, the control value is updated each time ϵ becomes negative. Usually, one tries to design an event-based feedback so that ϵ cannot remain negative (and so is updated the control only punctually). In addition, one also wants that two events are separated with a non vanishing time interval avoiding the *Zeno* phenomenon. All these properties are encompassed with the *Minimal inter-Sampling Interval* (MSI) property introduced in Marchand et al. (2013). In particular:

Property 1.2. (Semi-uniformly MSI).
An event-triggered feedback is said to be *semi-uniformly MSI* if and only if the inter-execution times can be below bounded by some non zero minimal sampling interval $\underline{\tau}(\delta) > 0$ for any $\delta > 0$ and any initial condition x_0 in the ball $\mathcal{B}(\delta)$ centered at the origin and of radius δ.

Remark 1.3. A semi-uniformly MSI event-driven control is a piecewise constant control with non zero sampling intervals (useful for implementation purpose).

A particular event-based feedback has been proposed in Marchand et al. (2013), based on the universal formula of Sontag (1989). In order to then understand how was built this strategy, we first recall some seminal results for the stabilization of continuous-time systems. Let consider the affine in the control nonlinear dynamical system

$$\begin{aligned} &\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t) \qquad (4)\\ &\text{with} \quad x(0) := x_0 \end{aligned}$$

where f and g are smooth functions with f vanishing at the origin.

Definition 1.4. (Control Lyapunov function).
A smooth and positive definite functional $V : \mathcal{X} \to \mathbb{R}$ is a *control Lyapunov function (CLF)* for system (4) if for each $x \neq 0$ there is some $u \in \mathcal{U}$ such that

$$\alpha(x) + \beta(x)u < 0 \qquad (5)$$

$$\text{with} \quad \begin{aligned} \alpha(x) &:= L_f V(x) = \frac{\partial V}{\partial x} f(x) \\ \beta(x) &:= L_g V(x) = \frac{\partial V}{\partial x} g(x) \end{aligned}$$

where $L_f V$ and $L_g V$ are the Lie derivatives of f and g functions respectively.

Property 1.5. (Small control property).
If for any $\mu > 0$, $\varepsilon > 0$ and x in the ball $\mathcal{B}(\mu)\backslash\{0\}$, there is some u with $\|u\| \le \varepsilon$ such that inequality (5) holds, then it is possible to design a feedback control that asymptotically stabilizes the system (Sontag (1989)).

Theorem 1.6. (Sontag's universal formula).
Assume that system (4) admits V as a CLF. For any real

analytic function $q : \mathbb{R} \to \mathbb{R}$ such that $q(0) = 0$ and $bq(b) > 0$ for $b \neq 0$, let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\phi(a,b) := \begin{cases} \dfrac{a + \sqrt{a^2 + bq(b)}}{b} & \text{if} \quad b \neq 0 \\ 0 & \text{if} \quad b = 0 \end{cases} \tag{6}$$

Then, the feedback $\upsilon : \mathcal{X} \to \mathcal{U}$, smooth on $\mathcal{X}^*$, defined by

$$\upsilon(x) := -\beta(x)\,\phi\Big(\alpha(x), \|\beta(x)\|^2\Big) \tag{7}$$

with $\alpha(x)$ and $\beta(x)$ defined in (5), is such that (5) is satisfied for all $x \in \mathcal{X}^*$.

Property 1.7. If the CLF V in Theorem 1.6 satisfies the *small control property*, then taking $q(b) = b$ in ϕ in (6), the control is continuous at the origin and so is globally asymptotically stable the closed-loop system.

The event-based feedback suggested in Marchand et al. (2013) is based on such an approach, where the control law υ is similar to the one in (7) (but with a lightly different function ϕ) and the event function ϵ is related to the time derivative of the CLF in order to ensure a (global) asymptotic stability of the closed-loop system. In this paper, we propose to extend this event-based feedback for the stabilization of nonlinear time-delay systems. Actually, the construction is quite similar, this is why we do not detail the event-based feedback for nonlinear undelayed systems here.

1.2 Stabilization of time-delay systems

Hereafter, the state of a time-delay system is described by $x_d : [-r, 0] \to \mathcal{X}$ defined by $x_d(t)(\theta) = x(t+\theta)$. This notation, used in Jankovic (2000) in particular, seems more convenient than the more conventional $x_t(\theta)$. Note that the dependence on t and θ can be omitted in the sequel for the sake of simplicity, writing $x_d(\theta)$ – or only x_d – instead of $x_d(t)(\theta)$ for instance. Let consider the affine in the control nonlinear dynamical time-delay system

$$\begin{aligned} &\dot{x} = f(x_d) + g(x_d)u \\ &\text{with} \quad x_d(0)(\theta) := \chi_0(\theta) \end{aligned} \tag{8}$$

where f, g are smooth functions and $\chi_0 : [-r, 0] \to \mathcal{X}$ is a given initial condition. Note that the class of time-delay system under consideration has been restricted to depend on l discrete delays and a distributed delay in the form

$$\begin{aligned} &\dot{x} = \Phi(x_\tau) + g(x_\tau)u \\ &\text{with} \quad \Phi(x_\tau) := f_0(x_\tau) + \int_{-r}^{0} \Gamma(\theta) F\big(x_\tau, x(t+\theta)\big) d\theta \\ &\text{and} \quad x_\tau := \big(x, x(t-\tau_1), \ldots, x(t-\tau_l)\big) \end{aligned} \tag{9}$$

where f_0, g and $F : \mathbb{R}^{(l+2)p} \to \mathbb{R}^{\Gamma}$ are smooth functions of their arguments. Without loss of generality, we assume that $F(x_\tau, 0) = 0$. The matrix $\Gamma : [-r, 0] \to \mathbb{R}^{p \times \Gamma}$ is assumed to be piecewise continuous (hence, integrable) and bounded.

Remark 1.8. The restriction (9) on this class of delay systems is needed to avoid the problems that arise due to non-compactness of closed bounded sets in the space $\big(C([-r, 0], \mathcal{X}), \|\cdot\|\big)$, where $C([-r, 0], \mathcal{X})$ denotes the space of continuous functions from $[-r, 0]$ into $\mathcal{X}$. This is discussed in Jankovic (1999, 2000).

Remark 1.9. Note that we do not consider input delays of the form $u(t-\tau)$ in this paper. However, the control law is computed using the state x_d of the time-delay system.

Definition 1.10. (Control Lyapunov-Krasovsky functional). A smooth functional $V : \mathcal{X} \to \mathbb{R}$ of the form

$$V(x_d) = V_1(x) + V_2(x_d) + V_3(x_d) \tag{10}$$

$$\text{with} \quad \begin{aligned} V_2(x_d) &= \sum_{j=1}^{l} \int_{-\tau_j}^{0} S_j(x(t-\varsigma)) d\varsigma \\ V_3(x_d) &= \int_{-r}^{0} \int_{t+\theta}^{t} L(\theta, x(\varsigma)) d\varsigma d\theta \end{aligned}$$

where V_1 is a smooth, positive definite, radially unbounded function of the current state x, V_2 and V_3 are nonnegative functionals respectively due to the discrete delays and the distributed delay in (9), $S_j : \mathcal{X} \to \mathbb{R}$ and $L : \mathbb{R}^+ \times \mathcal{X} \to \mathbb{R}$ are nonnegative integrable functions, smooth in the x-argument, is a *control Lyapunov-Krasovsky functional (CLKF)* for system (9) if there exist a function λ, with $\lambda(s) > 0$ for $s > 0$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2 such that

$$\kappa_1(|\chi_0|) \leq V(\chi_d) \leq \kappa_2(\|\chi_d\|)$$

and

$$\beta_d(\chi_d) = 0 \Rightarrow \alpha_d(\chi_d) \leq -\lambda(|\chi_0|) \tag{11}$$

$$\text{with} \quad \begin{aligned} \alpha_d(x_d) &:= L_f^* V(x_d) \\ \beta_d(x_d) &:= L_g V_1(x_d) \end{aligned}$$

for all piecewise continuous functions $\chi_d : [-r, 0] \to \mathcal{X}$, where χ_0 is defined in (8).

Remark 1.11. Whereas the classical Lie derivative notation is used in $L_g V_1(x) = \frac{\partial V_1}{\partial x} g(x)$ for the part function of the current state x, an extended Lie derivative is required for functionals of the form (10). $L_f^* V$, initially defined in Jankovic (2000), comes from the time derivative of the CLKF V in (10) along trajectories of the system (9), that is

$$\begin{aligned} \dot{V} &= \frac{\partial V_1}{\partial x}\Phi + \frac{\partial V_1}{\partial x} g u \\ &\quad + \sum_{j=1}^{l} \Big(S_j(x) - S_j(x(t-\tau_j))\Big) \\ &\quad + \int_{-r}^{0} \Big(L(\theta, x) - L(\theta, x(t+\theta))\Big) d\theta \\ &= L_f^* V(x_d) + L_g V_1(x_d) u \\ &= \alpha_d(x_d) + \beta_d(x_d) u \end{aligned} \tag{12}$$

when using the notation in (11), where Φ is defined in (9), which gives

$$\begin{aligned} L_f^* V(x_d) := L_\Phi V_1 &+ \sum_{j=1}^{l} \Big(S_j(x) - S_j(x(t-\tau_j))\Big) \\ &+ \int_{-r}^{0} \Big(L(\theta, x) - L(\theta, x(t+\theta))\Big) d\theta \end{aligned}$$

The Sontag's universal formula (Theorem 1.6) has been extended in Jankovic (2000) for the stabilization of nonlinear time-delay systems (9) with a CLKF of the form (10). This can be summarized as follows:

Theorem 1.12. (Sontag's universal formula with CLKF). Assume that system (9) admits a CLKF of the form (10). For any real analytic function $q : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$, both defined in Theorem 1.6, let the feedback $\upsilon : \mathcal{X} \to \mathcal{U}$, smooth on $\mathcal{X}^*$, defined by

$$\upsilon(x_d) := -\beta_d(x_\tau)\,\phi\Big(\alpha_d(x_d), \|\beta_d(x_d)\|^2\Big) \tag{13}$$

with x_τ and α_d, β_d defined in (9) and (11) respectively. Then υ is such that (11) is satisfied for all non zero piecewise continuous functions $\chi_d : [-r, 0] \to \mathcal{X}$.

Property 1.13. If the CLKF V in Theorem 1.12 satisfies the *small control property*, then taking $q(b) = b$ in ϕ in (6), the control is continuous at the origin and so is globally asymptotically stable the closed-loop system.

1.3 Contribution of the paper

In the present paper, we propose to extend the event-based approach previously developed in Marchand et al. (2013) for nonlinear time-delay systems admitting a CLKF.

In the sequel, let

$$x_{di} := x_d(t_i) \tag{14}$$

be the memory of the delayed state value at the last event, by analogy with (3).

2. EVENT-BASED STABILIZATION OF NONLINEAR TIME-DELAY SYSTEMS

It is possible to design an event-based feedback control that asymptotically stabilizes time-delay systems (9) with a CLKF of the form (10):

Theorem 2.1. (Event-based universal formula with CLKF). If there exists a CLKF V of the form (10) for system (9), then the event-based feedback (ϵ, υ) – see Definition 1.1 – defined by

$$\upsilon(x_d) = -\beta_d(x_\tau)\Delta(x_\tau)\gamma(x_d) \tag{15}$$

$$\begin{aligned}\epsilon(x_d, x_{di}) &= -\alpha_d(x_d) - \beta_d(x_d)\upsilon(x_{di}) \\ &\quad -\sigma\sqrt{\alpha_d(x_d)^2 + \theta(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\end{aligned} \tag{16}$$

with

- α_d and β_d as defined in (11)
- $\Delta : \mathcal{X}^* \to \mathbb{R}^{q\times q}$ and $\theta : \mathcal{X} \to \mathbb{R}$ are smooth positive definite functions
- $\gamma : \mathcal{X} \to \mathbb{R}$ is defined by

$$\gamma(x_d) := \begin{cases} \dfrac{\alpha_d(x_d) + \sqrt{\alpha_d(x_d)^2 + \theta(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} \\ \quad \text{if} \quad x_d \in \mathcal{S}_d \\ 0 \quad \text{if} \quad x_d \notin \mathcal{S}_d \end{cases} \tag{17}$$

- $\mathcal{S}_d := \{x_d \in \mathcal{X} \mid \|\beta_d(x_d)\| \neq 0\}$
- $\sigma \in [0, 1[$ is a tunable parameter

where x_{di} and x_τ are defined in (14) and (9) respectively, is semi-uniformly MSI, smooth on $\mathcal{X}^*$ and such that the time derivative of V satisfies (11) $\forall x \in \mathcal{X}^*$.

Remark 2.2. The simplification made with respect to the original result in Marchand et al. (2013) (for the stabilization of nonlinear undelayed systems) resides in the assumptions made for the functions θ and Δ, that are more restrictive here whereas they are assumed to be definite only on the set $\mathcal{S}_d$ in the original work.

Remark 2.3. The idea behind the construction of the event-based feedback (15)-(16) is to compare the time derivative of the CLKF V in the event-based case, that is applying $\upsilon(x_{di})$, and in the classical case, that is applying $\upsilon(x_d)$ instead of $\upsilon(x_{di})$. The event function is the weighted difference between both, where σ is the weighted value. By construction, an event is enforced when the event function ϵ vanishes to zero, that is hence when the stability of the event-based scheme does not behave as the one in the classical case. Also, the convergence will be faster with higher σ but with more frequent events in return. $\sigma = 0$ means updating the control when $\dot{V} = 0$.

Also, some properties are inherited from Marchand et al. (2013) and complete the Theorem 2.1. In particular:

Property 2.4. (Global asymptotic stability).
If the CLKF V in Theorem 2.1 satisfies the *small control property*, then the event-based feedback (15)-(16) is continuous at the origin and so is globally asymptotically stable the closed-loop system.

Property 2.5. (Smooth control).
If there exists some smooth function $\omega : \mathcal{X} \to \mathbb{R}^+$ such that on $\mathcal{S}_d^* := \mathcal{S}_d \backslash \{0\}$

$$\omega(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T - \alpha_d(x_d) > 0$$

then the control is smooth on $\mathcal{X}$ as soon as $\theta(x_d)\|\Delta(x_d)\|$ vanishes at the origin with

$$\begin{aligned}\theta(x_d) := {}& \omega(x_d)^2\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T \\ & - 2\alpha_d(x_d)\omega(x_d)\end{aligned} \tag{18}$$

2.1 Proofs

Proof of Theorem 2.1 The proof follows the one developed in Marchand et al. (2013) for event-based control of systems without delays (4). First, let define

$$\psi(x) := \sqrt{\alpha_d(x)^2 + \theta(x)\beta_d(x)\Delta(x)\beta_d(x)^T} \tag{19}$$

for the lake of simplicity in the sequel.

We begin establishing γ is smooth on $\mathcal{X}^*$. For this, consider the algebraic equation

$$\begin{aligned}P(x_d, \zeta) := {}& \beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T\zeta^2 \\ & - 2\alpha_d(x_d)\zeta - \theta(x_d) = 0\end{aligned} \tag{20}$$

Note first that $\zeta = \gamma(x)$ is a solution of (20) for all $x_d \in \mathcal{X}$. It is easy to prove that the partial derivative of P with respect to ζ is always strictly positive on $\mathcal{X}^*$

$$\frac{\partial P}{\partial \zeta} := 2\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T\zeta - 2\alpha_d(x_d) \tag{21}$$

Indeed, when $\|\beta_d(x_d)\| = 0$, (11) gives $\frac{\partial P}{\partial \zeta} = -2\alpha_d(x_d) \geq 2\lambda(|\chi_0|) > 0$ and when $\|\beta_d(x_d)\| \neq 0$, (17) gives $\frac{\partial P}{\partial \zeta} = 2\sqrt{\alpha_d(x_d)^2 + \theta(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} > 0$ replacing ζ in (21) by the expression of γ (since $\zeta = \gamma(x)$ is a solution of (20)). Therefore $\frac{\partial P}{\partial \zeta}$ never vanishes at each point of the form $\{(x_d, \gamma(x_d))|x_d \in \mathcal{X}^*\}$. Furthermore, P is smooth w.r.t. x_d and ζ since so are α_d, β_d, θ and Δ. Hence, using the implicit function theorem, γ is smooth on $\mathcal{X}^*$.

The decrease of the CLKF of the form (10) when applying the event-based feedback (15)-(16) is easy to prove. For this, let us consider the time interval $[t_i, t_{i+1}]$, that is the interval separating two successive events. Recall that x_{di} denotes the value of the state when the i^{th} event occurs and t_i the corresponding time instant, as defined in (14). At time t_i, when the event occurs, the time derivative of the CLKF, i.e. (12), after the update of the control is

$$\begin{aligned}\frac{dV}{dt}(x_{di}) &= \alpha_d(x_{di}) + \beta_d(x_{di})\upsilon(x_{di})\\ &= -\psi(x_{di}) < 0\end{aligned}$$

when substituting (17) in (15), where ψ is defined in (19). More precisely, defining a compact set not containing the origin, that is $\Omega = \{x_d \in CP([-r, 0], \mathcal{X}) : d \leq \|x_d\| \leq D\}$, where $CP([-r, 0], \mathcal{X})$ denotes the space of piecewise continuous functions from $[-r, 0]$ into $\mathcal{X}$, d and D are some constant in $\mathbb{R}^+$. If V is a CLKF for the system of the form (9) then for all $0 < \delta < D$ there exists $\varepsilon > 0$ such that $\alpha_d(\chi_d) \geq -\frac{1}{2}\lambda(|\chi_0|) \Rightarrow |\beta_d(\chi_d)| \geq \varepsilon$ for $\chi_d \in \Omega$. As a consequence, one obtains (see Lemma 1 in Jankovic (2000), and Jankovic (1999), for further details)

$$\dot{V} \leq -\lambda(|x|)$$

With this updated control, the event function (16) hence becomes strictly positive

$$\epsilon(x_{di}, x_{di}) = (1 - \sigma)\psi(x_{di}) > 0$$

since $\sigma \in [0, 1[$, where ψ is defined in (19). Furthermore, the event-function necessarily remains positive before the next event by continuity, because an event will occur when $\epsilon(x_d, x_{di}) = 0$ (see Definition 1.1). Therefore, on the interval $[t_i, t_{i+1}]$, one has

$$\begin{aligned}\epsilon(x_d, x_{di}) &= -\alpha_d(x_d) - \beta_d(x_d)\upsilon(x_{di}) - \sigma\psi(x_d)\\ &= -\frac{dV}{dt}(x_d) - \sigma\psi(x_d) \geq 0\end{aligned}$$

which ensures the decrease of the CLKF on the interval since $\sigma\psi(x_d) \geq 0$, where ψ is defined in (19). Moreover, t_{i+1} is necessarily bounded since, if not, V should converge to a constant value where $\frac{dV}{dt} = 0$, which is impossible thanks to the inequality above. The event function precisely prevents this phenomena detecting when $\frac{dV}{dt}$ is *close to vanish* and updates the control if it happens, where σ is a tunable parameter fixing how "close to vanish" has to be the time derivative of V.

To prove that the event-based control is MSI, we have to prove that for any initial condition in a priori given set, the sampling intervals are below bounded. First of all, notice that events only occur when ϵ becomes negative (with $x_d \neq 0$). Therefore, using the fact that when $\beta_d(x_d) = 0$, $\alpha_d(x_d) < -\lambda(|\chi_0|)$ (because V is a CLKF as defined in Definition 1.10), it follows from (16), on $\{x_d \in \mathcal{X}^* \mid \|\beta_d(x_d)\| = 0\}$, that

$$\epsilon(x_d, x_{di}) = -\alpha_d(x_d) - \sigma|\alpha_d(x_d)| = (1 - \sigma)\lambda(|\chi_0|) > 0$$

because $\sigma \in [0, 1[$ and $\lambda(s) > 0$ for $s > 0$. Therefore, there is no event on the set $\{x_d \in \mathcal{X} | \|\beta_d(x_d)\| = 0\} \cup \{0\}$. We then restrict the study to the set $\mathcal{S}_d^* = \{x_d \in \mathcal{X}^* | \|\beta_d(x_d)\| \neq 0\}$, where θ and Δ are strictly positive by assumption. Let us rewrite the time derivative of the CLKF along the trajectories, that is

$$\begin{aligned}\frac{dV}{dt}(x_d) &= \alpha_d(x_d) + \beta_d(x_d)\upsilon(x_{di})\\ &= -\psi(x_d) + \beta_d(x_d)\big(\upsilon(x_{di}) - \upsilon(x_d)\big)\end{aligned} \tag{22}$$

when using the definition of $\upsilon(x_d)$ in (15) and (17), where ψ is defined in (19). Let us define for $x_{di} \in \mathcal{S}_d$, the level $\vartheta_i := V(x_{di})$ and the set $\mathcal{V}_{\vartheta_i} := \{x_d \in \mathcal{X} | V(x_d) \leq \vartheta_i\}$. From the choice of the event function, it follows from (22) that x_d belongs to $\mathcal{V}_\vartheta \subset \mathcal{V}_{\vartheta_i}$. Note that if x_{di} belongs to $\mathcal{S}_d$, this is not necessarily the case for x_d that can escape from this set. First see that, since i) $\theta(x_d)$ is such that $\alpha_d(x_d)^2 + \theta(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T > 0$ for all $x_d \in \mathcal{S}_d^*$, and ii) $\alpha_d(x_d)$ is necessarily nonzero on the frontier of $\mathcal{S}_d$ (except possibly at the origin)

$$\begin{aligned}\frac{dV}{dt}(x_{di}) &= -\psi(x_{di})\\ &\leq -\inf_{\substack{x_{di} \in \mathcal{S}_d\\ \text{s.t. } V(x_{di}) = \vartheta_i}} \psi(x_{di}) =: -\varphi(\vartheta_i) < 0\end{aligned} \tag{23}$$

Considering now the second time derivative of the CLKF

$$\begin{aligned}\ddot{V}(x_d) &= \left(\frac{\partial \alpha_d}{\partial x_d}(x_d) + \upsilon(x_{di})^T \frac{\partial \beta_d^T}{\partial x_d}(x_d)\right)\Theta(x_d, x_{di})\\ \text{with} \quad &\Theta(x_d, x_{di}) := \Phi(x_\tau) + g(x_\tau)\upsilon(x_{di})\end{aligned} \tag{24}$$

where Φ is defined in (9). By continuity of all the involved functions (except for Γ in Φ which is piecewise continuous but bounded by assumption), both terms can be bounded for all $x_d \in \mathcal{V}_{\vartheta_i}$ by the following upper bounds $\varrho_1(\vartheta_i)$ and $\varrho_2(\vartheta_i)$ such that

$$\begin{aligned}\varrho_1(\vartheta_i) &:= \sup_{\substack{x_{di} \in \mathcal{S}_d \text{ s.t. } V(x_{di}) = \vartheta_i\\ x_d \in \mathcal{V}_{\vartheta_i}}} \left\| \frac{\partial \alpha_d}{\partial x_d}(x_d) + \upsilon(x_{di})^T \frac{\partial \beta_d^T}{\partial x_d}(x_d) \right\|\\ \varrho_2(\vartheta_i) &:= \sup_{\substack{x_{di} \in \mathcal{S}_d \text{ s.t. } V(x_{di}) = \vartheta_i\\ x_d \in \mathcal{V}_{\vartheta_i}}} \|\Theta(x_d, x_{di})\|\end{aligned}$$

where Θ is defined in (24). Therefore, $\dot{V}$ is strictly negative at any event instant t_i and cannot vanish until a certain time $\underline{\tau}(\vartheta_i)$ is elapsed (because its slope is positive). This minimal sampling interval is only depending on the level ϑ_i. A bound on $\underline{\tau}(\vartheta_i)$ is given by the inequality

$$\frac{dV}{dt}(x_d) \leq \frac{dV}{dt}(x_{di}) + \rho_1\rho_2(t - t_i) \qquad x \in \mathcal{V}_{\vartheta_i}$$

that yields

$$\underline{\tau}(\vartheta_i) \geq \frac{\varphi(\vartheta_i)}{\varrho_1(\vartheta_i)\varrho_2(\vartheta_i)} > 0$$

where φ is defined in (23). As a consequence, the event-based feedback (15)-(16) is semi-uniformly MSI.

Proof of Property 2.4 To prove the continuity of υ at the origin, one only needs to consider the points in $\mathcal{S}$ since we already have $\upsilon(x_d) = 0$ if $\|\beta_d(x_d)\| = 0$. From (15), we have

$$\begin{aligned} \|\upsilon(x_d)\| &\leq \frac{|\alpha_d(x_d)|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\| \\ &\quad + \frac{\psi(x_d)}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\| \\ &\leq \frac{2|\alpha_d(x_d)|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\| \\ &\quad + \sqrt{\theta(x_d)\|\Delta(x_d)\|} \end{aligned} \tag{25}$$

With the small control property (see Property 1.5), for any $\varepsilon > 0$, there is $\mu > 0$ such that for any $x_d \in \mathcal{B}(\mu)\backslash\{0\}$, there exists some u with $\|u\| \leq \varepsilon$ such that $L_f^* V(x_d) + [L_g V_1(x_d)]^T u = \alpha_d(x_d) + \beta_d(x_d)u < 0$ and therefore $|\alpha_d(x_d)| < \|\beta_d(x_d)\|\varepsilon$. It follows:

$$\begin{aligned} \|\upsilon(x_d)\| \leq & \frac{2\varepsilon\|\beta_d(x_d)\|\|\Delta(x_d)\beta_d(x_d)^T\|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} \\ & + \sqrt{\theta(x_d)\|\Delta(x_d)\|} \end{aligned}$$

Since the function $(v_1, v_2) \to \frac{\|v_1\|\|v_2\|}{v_1^T v_2}$ is continuous with respect to its two variables at the origin where it equals 1, since θ and Δ are also continuous, since $\theta(x_d)\|\Delta(x_d)\|$ vanishes at the origin, for any ε', there is some μ' such that $\forall x_d \in \mathcal{B}(\mu')\backslash\{0\}$, $\|\upsilon(x_d)\| \leq \varepsilon'$ which ends the proof of continuity.

Proof of Property 2.5 Finally, with θ as in (18), the control becomes $\upsilon(x_d) = -\beta_d(x_d)\Delta(x_d)\omega(x_d)$ which is obviously smooth on $\mathcal{X}$.

2.2 Example

Consider the nonlinear time-delay system

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_2 + x_{2d} + x_1^3 + u \\ \text{with} \quad x_{2d} &:= x_2(t - \tau) \end{aligned} \tag{26}$$

that admits a CLKF (proposed in Jankovic (2000))

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}\int_{-\tau}^{0} x_2^2(\theta)d\theta \tag{27}$$

$$\text{with} \quad \begin{aligned} \alpha_d &= x_2(-x_2 + x_{2d} + x_1^3) + \frac{1}{2}(x_2^2 - x_{2d}^2) \\ \beta_d &= x_1 + x_2 \end{aligned}$$

Indeed, setting, $\lambda(|x|) = \frac{1}{4}|x|^4$, one obtains

$$\begin{aligned} &\beta_d = 0 \Rightarrow x_1 = -x_2 \\ &\Rightarrow \alpha_d = -\frac{1}{2}(x_2 - x_{2d})^2 - x_2^4 \leq -x_2^4 \leq -\lambda(|x|) \end{aligned}$$

which proves that (27) is a CLKF for (26) using Definition 1.10.

The time evolution of x, $\upsilon(x)$ and the event function $\epsilon(x, x_i)$ is depicted in Fig. 1, for $\Delta = I_p$ (the identity matrix), $\theta(x)$ is as defined in (18) (for smooth control everywhere), with $\omega = 0.1$, $\sigma = 0.1$, $x_0 = (1\ -2)^T$ and a time delay $\tau = 2\,s$. One could remark that only 5 events occurs in the $20\,s$ simulation time (including the first event at $t = 0$) when applying the proposed event-based approach (15)-(16).

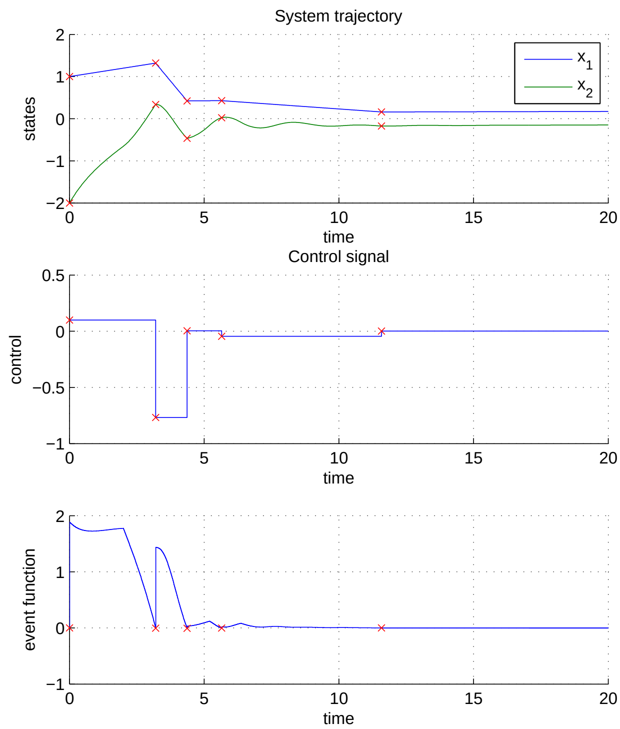

Fig. 1. Sumulation results of system (26) with CLKF as in (27) and event-based feedback (15)-(16).

CONCLUSION

In this paper, we proposed an extension of the *Sontag*'s universal formula for event-based stabilization of nonlinear *time-delay* systems. Whereas the original work deals with control Lyapunov functions, some *control Lyapunov-Krosovsky functionals* (CLKF) are now required for a global (except at the origin) asymptotic stabilization of systems with state delays. The sampling intervals do not contract to zero. Moreover, as in the original result, if the CLKF fulfills the small control property then the control is continuous at the origin. With additional assumption, the control can be proved to be smooth everywhere. Some simulation results were provided, they notably highlighted the low frequency of events of the proposal.

Next step is to also consider input delays. Another way of investigation could be to develop event-based strategies for nonlinear systems based on other universal formulas, like the formula of Freeman and Kokotovic (1996) or the domination redesign formula of Sepulchre et al. (1997), using CLRF and CLKF in the spirit of Jankovic (2000) (for the time-triggered case).

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